Introduction to suffix notation

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Suffix notation can be a frequent source of confusion at first, but it is a useful tool for manipulating matrices. We will use the convention that if A is a matrix, then $(A)_{ij} = a_{ij}$ is the element of that matrix in the ith row and jth column. Suffix notation becomes especially important when one deals with tensors, which can be thought of as the generalisation of familiar objects - scalars (0 dimension), vectors (1 dimension), matrices (2 dimensions) - to higher dimension. Even when not dealing with tensors, however, suffix notation is a useful thing to understand.

We will begin by reviewing why matrix multiplication works the way it does. One way of thinking of vector equations is as a shorthand for a set of simultaneous equations - each component of the vectors gives an equation. Explicitly, consider the set of three equations for the three unknowns x_1, x_2, x_3:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \tag{1}$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \tag{2}$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \tag{3}$$

This can be rewritten in a matrix/vector form as equation $Ax = b$:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{4}$$

Comparison of these two forms should convince you that the “go along the column and down the rows” rule for multiplying a matrix and a vector is sensible. We can also write equations 1-3 more succintly in suffix notation. We notice that in any of the three equations, the first index on the a_{ij} elements is fixed whilst the second varies from 1 to 3. Thus:

$$\sum_{j=1}^{3} a_{1j}x_j = b_1 \tag{5}$$
$$\sum_{j=1}^{3} a_{2j}x_j = b_2 \tag{6}$$
$$\sum_{j=1}^{3} a_{3j}x_j = b_3 \tag{7}$$

Even more succintly, we can write this as the single expression

$$\sum_{j=1}^{3} a_{ij}x_j = b_i \tag{8}$$

When you see such an equation, remember that it is a shorthand notation for writing three equations at once, for $i = 1, 2, 3$ (in 3D). Next, consider the product of two matrices, PQ. One way of thinking of a matrix is as a series of vectors, so let us write the matrix P as the three vectors (q_1, q_2, q_3). We form the matrix/vector products Pq_1, Pq_2, Pq_3 to give three new vectors. We could then put together to form a new matrix, which will just be the product PQ.

We can instead use suffix notation to see why matrix multiplication must work as it does. Consider first forming the product of two matrices, AB, which is itself a matrix. Then form the product ABx. Matrix multiplication is associative, so we can consider this as either $(AB)x$ or $A(Bx)$. In suffix notation, using Eqn. 8 for the product of the matrix B with vector x, or for the product of matrix A with vector Bx:

$$\sum_j (AB)_{ij} x_j = \sum_k A_{ik} (Bx)_k = \sum_{j,k} A_{ik} B_{kj} x_j \tag{9}$$

The vector x is arbitary, so we can therefore deduce the rule for finding the product of two matrices:

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} \tag{10}$$

When writing down equations involving suffices, you must make sure that every term has the correct number of indices. It is incorrect to write $Ax = \sum_j a_{ij} x_j$: the left hand side is a vector, whereas the right hand side is a component of that vector. You should instead write $(Ax)_i = \sum_j a_{ij} x_j$. This equation illustrates the two types of suffices we have. If a suffix appears once on each term in an equation, it is a *free* index, and must appear exactly once on every term. If a suffix appears twice, it is a *dummy* index and will be summed over (When dealing with complicated expressions one often uses the summation convention, which is that any index appearing twice is automatically summed over and you don't write the Σ. For example, Eqn. 8 would be just $a_{ij} x_j = b_i$). If you have an expression with an index appearing more than twice, it is wrong.

You are free to relabel a dummy index to anything you choose, for example $\sum_j a_{ij} x_j = \sum_k a_{ik} x_k$. (This is analogous to renaming variables that are being integrated, such as $\int x \, \mathrm{d}x = \int y \, \mathrm{d}y$.) Consequently, when you write down an expression involving the product of many matrices, make sure that you choose a different dummy index to sum over for each of the products. For example, the product of four matrices $ABCD$ is

$$(ABCD)_{ij} = \sum_l \sum_m \sum_n a_{il} b_{lm} c_{mn} d_{nj} \tag{11}$$

Also notice that although matrix multiplication does not commute ($AB \neq BA$ except in special cases), the objects in the right hand of the sum (11) are just ordinary numbers being multiplied together, so we could write them in any order we choose, such as

$$(ABCD)_{ij} = \sum_l \sum_m \sum_n c_{mn} b_{lm} a_{il} d_{nj} = \sum_l \sum_m \sum_n a_{il} d_{nj} c_{mn} b_{lm} \tag{12}$$

It is not, however, immediately obvious what the right hand side of Eqn. 12 represents, so it is generally best to ensure that any repeated indices are kept next to each other, as in Eqn. 11.

We finish by mentioning two special objects you have encountered, the Kronecker delta (δ_{ij}) and the Levi-Civita symbol (ϵ_{ijk}). The Kronecker delta is defined as

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and can be used to select elements from a vector. To see this, note that from the above definition $\sum_j \delta_{ij} x_j = x_i$. It can also be used to concisely express that a set of basis vectors is orthonormal: $\boldsymbol{x}_i \cdot \boldsymbol{x}_j = \delta_{ij}$. Note that this definition of the Kronecker hold regardless of what dimension we are working in: i and j range from 1 to N for whatever value of N is appropriate. The Levi-Civita symbol, however, is defined as acting on three dimensional vectors and matrices (though a similar object can be defined in more than three dimensions). Its definition is

$$\epsilon_{ijk} = \begin{cases} +1 & i,j,k \text{ are a cyclic permutation of 1,2,3} \\ -1 & i,j,k \text{ are an anticyclic permutation of 1,2,3} \\ 0 & \text{if any of the indices are equal} \end{cases}$$

Of the 27 possible index combinations, there are therefore only 6 that are non-zero: $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$ and $\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$. This allows us to simply write an expression for the cross product of two vectors:

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \sum_{j,k} \epsilon_{ijk} a_j b_k \tag{13}$$

Taking the 1 component as an example, the right hand side is then non-zero for $j = 2, k = 3$ and $j = 3, k = 2$ which means the Levi-Civita symbol takes values +1 and -1 respectively. Thus, $(\boldsymbol{a} \times \boldsymbol{b})_1 = a_2 b_3 - a_3 b_2$, as expected. In a similar fashion, we can write an expression for the determinant of a 3×3 matrix using ϵ_{ijk}:

$$|A| \epsilon_{lmn} = \sum_{i,j,k} a_{li} a_{mj} a_{nk} \epsilon_{ijk} \tag{14}$$

For example, setting l, m, n equal to 1,2,3

$$|A| = \sum_{i,j,k} a_{1i} a_{2j} a_{3k} \epsilon_{ijk} \tag{15}$$

If you write this expression out explicitly you will see it is identical to performing a Laplace expansion along the first row of a matrix. Eqn.14 illustrates a number of properties of determinants, such as the fact that swapping two rows or columns changes the sign of the determinant (because ϵ_{ijk} must change between being a cyclic and an anti-cyclic permutation).

Tensors (not for IA)

Vectors and matrices are examples of more general objects called tensors. Tensors are defined via their transformation properties: suppose we have a set of three numbers v_i (we'll assume 3D, but generalising to higher dimension is straightforward), and we want to know how their values change under rotation of Cartesian axes. If the values in the new co-ordinate system v'_i can be written

$$v'_i = L_{ij} v_j \tag{16}$$

then the v_i are said to be the components of a rank one tensor. (Although L_{ij} will be the components of a matrix, for current purposes it is perhaps best for now to think of it as just being a set of 9 numbers such that the above equation is true.) Similarly, the components of a rank two tensor satisfy

$$a'_{ij} = L_{im} L_{jn} a_{mn} \tag{17}$$

and for higher order tensors, we just keep adding more of the L_{ij} rotation matrices. What you have previously called scalars, vectors and matrices are in fact rank zero, rank one and rank two tensors respectively.

Rotations are described by orthogonal matrices: $LL^T = I$. Thus, $|L| = \pm 1$. The rotations you have met so far will generally have had $|L| = +1$; these are called proper rotations. If $|L| = -1$ the rotation is called improper: geometrically, as well as rotation the matrix also reflects the co-ordinate system through the origin. Thus, if a set of numbers v_i satisfies the transformation law $v'_i = L_{ij} v_j$ for *all* L (both proper and improper) then the v_i form a tensor. However, if this only holds for proper rotations and instead $v'_i = -L_{ij} v_j$ for improper rotations the v_i are said to be the components of a pseudotensor. You are already familiar with an example of a pseudovector: any vector $\boldsymbol{c}$ such that $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ is a pseudovector, because under inversion of the co-ordinate system ($\boldsymbol{a} \to -\boldsymbol{a}, \boldsymbol{b} \to -\boldsymbol{b}$) the vector $\boldsymbol{c}$ is unchanged. An alternative way of thinking of this is to note that $(\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a_j b_k$ is only true in a right-handed co-ordinate system. If we chose to use a left-handed co-ordinate system we would have to introduce an extra minus sign somewhere to get the same physical vector as in the right-handed co-ordinates.

Also, the Levi-Civita symbol is in fact a pseudotensor: from the earlier discussion of using this symbol to find determinants,

$$\begin{aligned} |L|\epsilon_{ijk} &= L_{il} L_{jm} L_{kn} \epsilon_{lmn} &(18)\\ \epsilon_{ijk} &= |L| L_{il} L_{jm} L_{kn} \epsilon_{lmn} &(19) \end{aligned}$$

where we have used the fact that $|L| = \pm 1$. Thus, under an improper rotation the sign of ϵ_{ijk} changes and so it is a pseudotensor, as claimed.